Complexity of Concurrent Reachability Games: Approximation of the value

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Why do *I* care about *this* problem?

- Complexity may have diverse reactions
- People want to show problems are easy/hard
- My first work in the polynomial hierarchy
- Motivated Kristoffer Hansen to do research for life

History

- 2004: PhD thesis claimed to solve the problem
- 2008: Outlined proof was proven incomplete
- 2009: Example showed that the proof was wrong
- 2011: Alternative approach was proven to insufficient
- 2013: Weaker result than the claimed solution was proven
- 2017: Characterization of how far from true the claim was
- 2023?: The original claim will be proven to be true

Purgatory($n = 7,\ m = 9$)

Consider the following repeated game.

- Lucifer and Dante give a number in $\{1, 2, \ldots, 9\}$
- If Dante's number is higher than Lucifer's, Dante goes to hell
- If both numbers coincide 7 times in a row, Dante wins
- In any other case, they keep playing

Concurrent Reachability Games

A Concurrent Reachability Stochastic Game is a two-player game denoted $G = (V, \mathcal{A}_1, \mathcal{A}_2, \delta)$ consisting of

- the set of vertices V
- the target vertex $\mathbb{1} \in V$
- action sets for each player $\mathcal{A}_1$ and $\mathcal{A}_2$
- the transition function $\delta : V \times \mathcal{A}_1 \times \mathcal{A}_2 \to \Delta(V)$

Concurrent Reachability Games

The value is defined by

$$\mathrm{val}(v) := \sup_{\sigma} \inf_{\tau} \mathbb{P}_v^{(\sigma,\tau)}(\exists i \geq 1 \quad V_i = \mathbb{1}) \,.$$

Remark

There exist stationary ϵ-optimal strategies for both players.

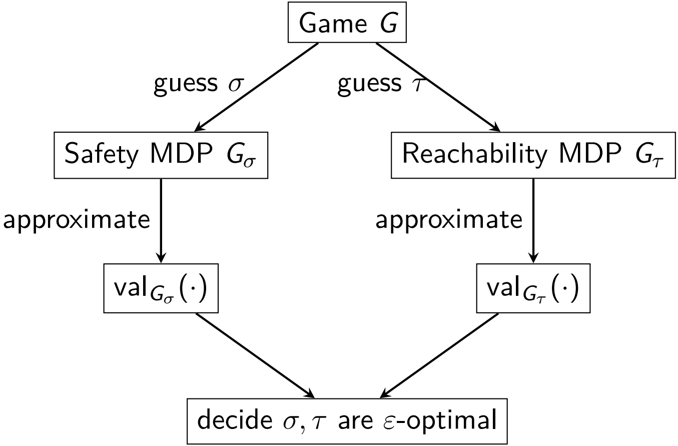
Definition (Approximation of the value)

Given $\varepsilon > 0$ and a game G, compute the value vector up to ε.

History: Approximating the value

- 2004: Claimed complexity TFNP, guessing strategies
- 2008: Outlined proof was proven incomplete
- 2009: ε-optimal strategies need exponential space
- 2011: Value iteration takes doubly exponential time
- 2013: Complexity TFNP[NP] was proven
- 2017: Characterization of complexity of ε-optimal strategies
- 2023?: Complexity TFNP will be proven

Sketch proof: Claim



How wrong could this proof be?

Hardness of strategies

Theorem (Required patience [HIM11, Theorem 10])

Suppose n is sufficiently large and $m \geq 2$. Let $\varepsilon = 1 - 4m^{-n/2}$. Then, all ε-optimal strategies of Purgatory(n, m) have patience at least $2^{m^{n/3}}$.

In other words, ε-optimal strategies require exponential space in binary representation.

Remark

For the Purgatory($n = 7$, $m = 9$), we have that $2^{m^{n/3}} \geq 2^{168}$.

Value iteration

- Assign value 0 to all states.
- Assign value 1 the target $\mathbb{1}$.
- In each step, update the value of a state, assuming continuation values.

Remark

Value iteration is known to take at least exponential time, even in Markov Chains.

Hardness of value iteration

Theorem (Required steps of value iteration [HIM11, Corollary 9])

Let n be even. Applying less than $2^{m^{n/2}}$ iterations of the value iteration algorithm to Purgatory(n, m) yields a valuation of the initial position of at most $3m^{-n/2}$, even though the actual value of the game is 1.

In other words, ε-optimal values can be computed through Value Iteration only after *doubly exponentially* many steps.

Remark

For the Purgatory($n = 7$, $m = 9$), we have that $2^{m^{n/2}} = 2^{2187}$.

Can we do anything?

Best upper bound so far: TFNP[NP]

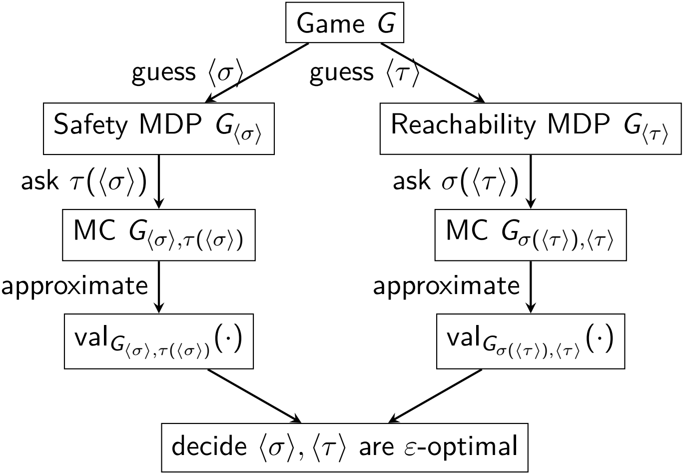
Theorem ([FM13, Theorem 1])

Approximating the value of Concurrent Reachability games has

- *polynomial size guess*
- *polynomial time verifier using an oracle of a problem in NP*

In other words, the problem is in TFNP[NP].

Sketch proof



Floating point

Consider a non-negative dyadic rational number of the form

$$m2^{-e},$$

where $m \in \{2^{b-1}, 2^{b-1}+1, 2^{b-1}+2, \ldots, 2^b - 1\}$ for some $b \geq 1$. The floating point encoding is $\langle 1^b, bit(m), bit(e) \rangle$, which is unique for a fixed precision b. Denote the set of floating points with precision b by $\mathbb{F}(b)$.

Examples

- 1, with encoding size 3
- 2^{-100}, with encoding size 9
- $1 - 2^{-100}$, with encoding size 108

Floating point: examples

For example,

$$1 \in \mathbb{F}(b = 1) \subseteq \mathbb{F}(b = 2) \subseteq \dots ,$$

and $\langle 1^{b=1}, bit(m = 1), bit(e = 0) \rangle = 1 + 1 + 1 = 3$. Also,

$$2^{-100} \in \mathbb{F}(b = 1) \,,$$

and $\langle 1^{b=1}, bit(m = 1), bit(e = 100) \rangle = 1 + 1 + \lceil \log(100) \rceil = 9$.
Lastly,

$$1 - 2^{-100} = (2^{100} - 1) 2^{-100} \in \mathbb{F}(b = 2^{100}) \,,$$

but
$\langle 1^{1}, bit(1 - 2^{100}), bit(100) \rangle = 1 + \lceil \log(2^{100}) \rceil + \lceil \log(100) \rceil = 108$.

Floating point: probabilities

Definition (Floating point probabilities)

For $b \geq 1$, define the set of probability measures represented by floating point weights by

$$\mathbb{P}[b] := \left\{ p \in \Delta([n]) : n \in \mathbb{N} \quad \exists w \in \mathbb{F}(b)^n \quad p_k = \frac{w_k}{\sum_{\ell \in [n]} w_\ell} \right\} .$$

Definition (Closeness)

We say that $x, y \in [0, 1]$ are (b, k)-close if

$$\delta(x, y) := \frac{\max(x, y)}{\min(x, y)} - 1 \leq \left(\frac{1}{1 - 2^{-b+1}} \right)^k - 1 .$$

Floating point: representability

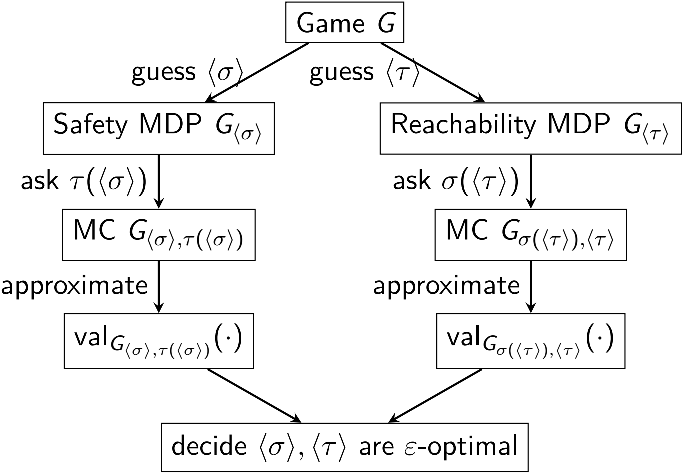
Lemma

Consider a probability distribution $q \in \Delta([k])$*. Then, there exists a probability distribution* $p \in \mathbb{P}[b]$ *such that*

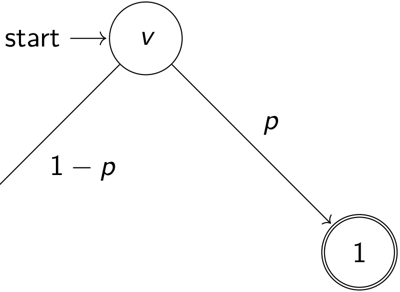
$$\delta(q, p) \leq \left(\frac{1}{1 - 2^{-b+1}} \right)^{2k+2} - 1 \,,$$

i.e. they are $(b, 2k+2)$*-close.*

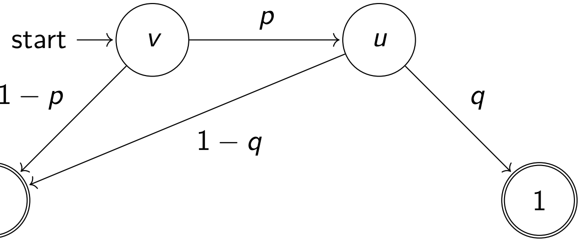
Sketch proof



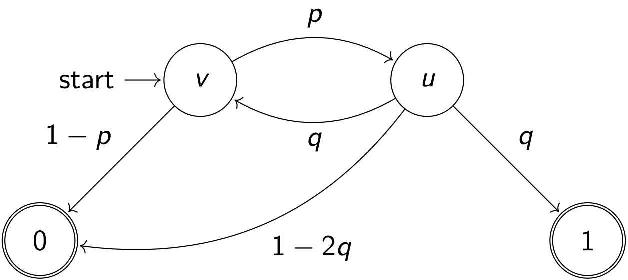
Reachability for Markov Chains



Reachability for Markov Chains



Reachability for Markov Chains



Reachability for Markov Chains: Algorithm

Theorem ([FM13, Theorem 4])

There is a polynomial time algorithm that takes input

- *Markov Chain with n states*
- *with transition probabilities in $\mathbb{P}[b]$, with $b \geq 1000n^2$*

operates with only multiplication, addition and division and outputs reachability probabilities with error at most $80n^4 2^{-b}$.

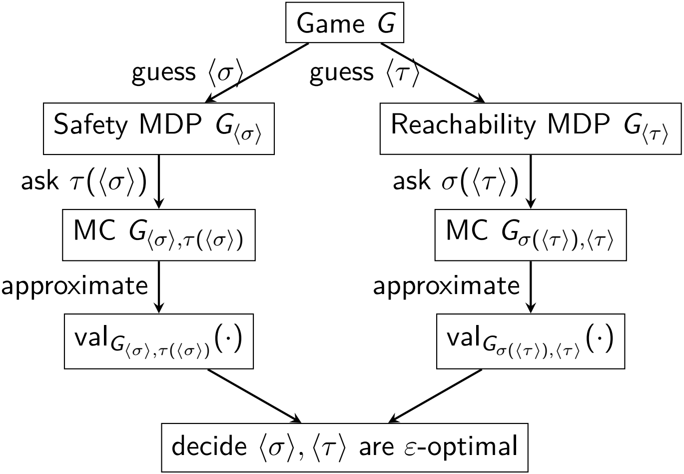
Concurrent Reachability Games: Existence of guess

Theorem ([CHI17, Theorem 14])

Consider a concurrent reachability game with n states, m actions per state, and transition probabilities that are rational numbers defined using at most B bits. Then, for all $\varepsilon > 0$, both players have an ε-optimal stationary strategy with denominators of at most

$$\frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right) 2^{nBm^{O(n^2)}} .$$

Sketch proof



Can we get rid of the oracle?

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